

Non-abelian free groups admit non-essentially free actions on rooted trees

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Abstract

We show that every finitely generated non-abelian free group Γ admits a spherically transitive action on a rooted tree T such that the action of Γ on the boundary of T is not essentially free. This reproves a result of Bergeron and Gaboriau.

The existence of such an action answers a question of Grigorchuk, Nekrashevich and Sushchanskii.

1 Introduction

Let Γ be a finitely generated group. A *chain* in Γ is a sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of subgroups of finite index in Γ . Let $T = T(\Gamma, (\Gamma_i))$ denote the *coset tree*, a rooted tree on the set of right cosets of the subgroups Γ_n defined by inclusion. The group Γ acts by automorphisms on T ; this action extends to the boundary ∂T of T , the set of infinite rays starting from the root. The boundary is naturally endowed with the product measure coming from the tree and Γ acts by measure-preserving homeomorphisms on ∂T . We call this action the *boundary representation* of Γ with respect to the chain (Γ_n) . It is easy to see that this action is always ergodic and minimal.

We say that the action of Γ on $\partial T(\Gamma, (\Gamma_i))$ is essentially free, if almost every element of $\partial T(\Gamma, (\Gamma_i))$ has trivial stabilizer in Γ . This is the case e.g. when the chain consists of normal subgroups of Γ and their intersection is trivial.

The main aim of this note is to construct faithful non-essentially free boundary representations of non-abelian free groups. Let F_d denote the free group of rank d .

Theorem 1 *Let $d \geq 2$ and let $\Gamma = F_d$. Then there exists a chain (Γ_n) in Γ such that $\cap \Gamma_n = 1$ and the boundary representation of Γ with respect to the chain (Γ_n) is not essentially free.*

This allows one to answer the question [4, Problem 7.3.3]. We say that two graphs are locally isomorphic if you can not distinguish them using just local information, that is, the isomorphism classes of finite balls coincide for the two graphs.

Problem 2 (Grigorchuk, Nekrashevich and Sushchanskii) *Does there exist a spherically transitive group of automorphisms of a rooted tree such that the Schreier graph of orbits, on the boundary, which are typical in the sense of Baire category (the orbits of generic points), are different from (not locally isomorphic to) the Schreier graph of orbits, on the boundary, which are typical in the sense of measure?*

Indeed, it turns out that for our action of F_d , the Schreier graph of an orbit of a Baire generic point is a $2d$ -regular tree, while for a measure generic point it is not.

Remark. The authors got aware that Theorem 1 has been proved by Bergeron and Gaboriau (see [2, Theorem 4.1, point 5])). Although the method of our proof is different, the result is the same and therefore we shall leave this paper as an expository article.

2 Proofs

We start with some definitions. Let (Γ_n) be a chain in Γ . Then the *coset tree* $T = T(\Gamma, (\Gamma_n))$ of Γ with respect to (Γ_n) is defined as follows. The vertex set of T equals

$$T = \{\Gamma_n g \mid n \geq 0, g \in \Gamma\}$$

and the edge set is defined by inclusion, that is,

$$(\Gamma_n g, \Gamma_m h) \text{ is an edge in } T \text{ if } m = n + 1 \text{ and } \Gamma_n g \supseteq \Gamma_m h.$$

Then T is a tree rooted at the vertex Γ and every vertex of level n has the same number of children, equal to the index $|\Gamma_n : \Gamma_{n+1}|$. The right actions of Γ on the coset spaces Γ/Γ_n respect the tree structure and so Γ acts on T by automorphisms. This action is called the *tree representation* of Γ with respect to (Γ_n) .

The boundary ∂T of T is defined as the set of infinite rays starting from the root. The boundary is naturally endowed with the product topology and product measure coming from the tree. More precisely, for $t = \Gamma_n g \in T$ let us define the *shadow* of t as

$$\text{Sh}(t) = \{x \in \partial T \mid t \in x\}$$

the set of rays going through t . Set the base of topology on ∂T to be the set of shadows and set the measure of a shadow to be

$$\mu(\text{Sh}(t)) = 1/|\Gamma : \Gamma_n|.$$

This turns ∂T into a totally disconnected compact space with a Borel probability measure μ . The group Γ acts on ∂T by measure-preserving homeomorphisms; we call this action the *boundary representation* of Γ with respect to (Γ_n) .

There are various levels of faithfulness of a boundary representation. Let

$$\partial T_{free} = \{x \in \partial T \mid \text{Stab}_\Gamma(x) = 1\}.$$

We say that the action is *free*, if $\partial T_{free} = \partial T$. The action is *essentially free* (or that the chain satisfies the Farber condition), if $\mu(\partial T \setminus \partial T_{free}) = 0$. The action is *topologically free* if $\partial T \setminus \partial T_{free}$ is meager, i.e., a countable union of nowhere dense closed sets. Note that the Farber condition has been introduced by Farber in [3] in another equivalent formulation (the name ‘Farber condition’ is from [2]).

It is easy to see that the following implications hold for a boundary representation of a countable group Γ :

$$\begin{aligned} (\Gamma_n) \text{ is normal and } (\cap \Gamma_n = 1) &\implies \text{free} \implies \text{essentially free} \implies \\ &\implies \text{topologically free} \iff \partial T_{free} \neq \emptyset \implies \text{faithful} \end{aligned}$$

For all but the third arrow it is easy to find examples showing that the reverse implications do not hold.

Now we will start building towards Theorem 1. The first lemma is straightforward from the definitions above.

Lemma 3 *Let Γ be a countable group with a chain (Γ_n) . Let*

$$\text{fixr}(g, \Gamma/\Gamma_n) = \frac{|\{x \in \Gamma/\Gamma_n \mid xg = x\}|}{|\Gamma : \Gamma_n|}$$

denote the ratio of fixed points of g acting on the right coset Γ/Γ_n . Then the boundary representation of Γ with respect to (Γ_n) is essentially free if and only if for all $g \in \Gamma$ with $g \neq 1$, the limit

$$\lim_{n \rightarrow \infty} \text{fixr}(g, \Gamma/\Gamma_n) = 0.$$

Let X be a set of symbols such that for all $x \in X$ the symbol $x^{-1} \notin X$. By an X -labeled graph $G = (V, E, l)$ we mean a finite, directed connected graph with vertex set V and edge set E together with a labeling function $l : E \rightarrow X$ that satisfies the following:

for all $v \in V, x \in X$ there is at most one $e \in E$ starting at v such that $l(e) = x$.

and

for all $v \in V, x \in X$ there is at most one $e \in E$ ending at v such that $l(e) = x$.

Note that we allow multiple edges and loops in G .

Let $G = (V, E, l)$ be an X -labeled graph. Then for every $x \in X$ we can associate a function $f_x : V \rightarrow V$ as follows. Let $G' = (V, E')$ be the directed

graph obtained by erasing all edges from G that is not x -labeled. For $v \in V$ let $C(v)$ denote the connected component of v in G' . Then $C(v)$ is either an isolated point, a directed circle or a directed simple path. If v is an isolated point then let $f_x(v) = v$. If $C(v)$ is a directed circle then let $f_x(v) = v'$ where $(v, v') \in E'$. Similarly, if $C(v)$ is a path and v is not the last point of $C(v)$ then let $f_x(v) = v'$ where $(v, v') \in E'$. Finally, if $C(v)$ is a path and v is the last point of $C(v)$ then let $f_x(v) = v'$ where v' is the first point of $C(v)$.

It is easy to see that f_x is a bijection of V for all $x \in X$. Let F_X denote the free group generated by the alphabet X . Then the mapping

$$\Phi : x \mapsto f_x$$

extends to a homomorphism from F_X to the symmetric group $\text{Sym}(V)$, that is, a permutation action of F_X on V . We will use the following property of Φ – the proof is a straightforward induction.

Lemma 4 *Let $w \in F_X$ be a reduced word of length k . Write*

$$w = u_1 u_2 \cdots u_k$$

where $u_i \in X$ or $u_i^{-1} \in X$. Let $v_0 \in V$ and for $i \geq 1$ let us define recursively $v_i \in V$ to be the vertex that satisfies

$$\begin{aligned} l(v_{i-1}, v_i) &= u_i & \text{if } u_i \in X \\ l(v_i, v_{i-1}) &= u_i^{-1} & \text{if } u_i^{-1} \in X \end{aligned}$$

Assume that the above recursive definition makes sense. Then

$$v_k = v_0 \Phi(w).$$

We are ready to prove Theorem 1.

Proof of Theorem 1.

Let

$$X = \{a, b, c_1, \dots, c_{d-2}\}$$

be an alphabet of d letters. Let $\Gamma = F_X$ be the free group on X . Let C be the set of conjugacy classes in Γ and let

$$C' = \{t \in C \mid \text{there is } w \in t \text{ starting with } a\}$$

Let us list the elements of C' as t_1, t_2, \dots and let $w_i \in t_i$ be a representative that starts with a ($i \geq 1$). Let $k_i = |w_i|$ be the length of w_i ($i \geq 1$) and let us decompose

$$w_i = u_{i,1} u_{i,2} \cdots u_{i,k_i}$$

where $u_{i,j} \in X$ or $u_{i,j}^{-1} \in X$ ($1 \leq j \leq k_i$). We can assume that the sequence k_i is non-decreasing. Let $0 < \alpha < 1$ and let p_i be an increasing sequence of prime numbers satisfying

$$\prod_{i=1}^n \left(1 - \frac{k_i + 1}{p_i + k_i}\right) > \alpha$$

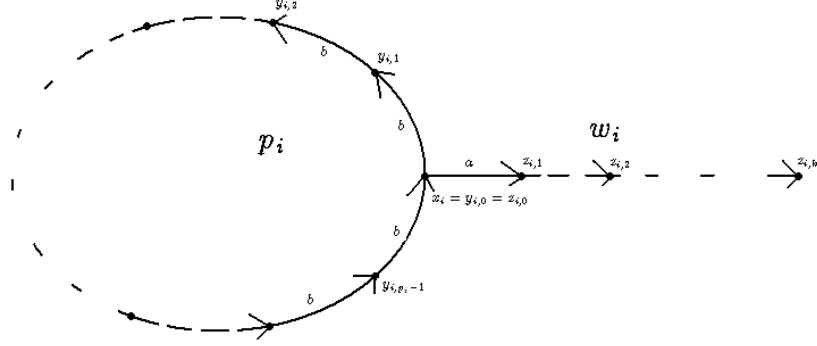


Figure 1: The graph G_i

for all $n \geq 1$.

For $i \geq 1$ let $G_i = (V_i, E_i, l_i)$ be an X -labeled graph defined as follows. Let

$$Y_i = \{y_{i,0}, y_{i,1}, \dots, y_{i,p_i-1}\} \text{ and } Z_i = \{z_{i,0}, z_{i,1}, \dots, z_{i,k_i}\}$$

be sets such that $x_i = y_{i,0} = z_{i,0}$ and $Y_i \cap Z_i = \{x_i\}$. For convenience, denote $y_{i,p_i} = y_{i,0}$. Let $V_i = Y_i \cup Z_i$ and assume that $V_i \cap V_j = \emptyset$ ($i \neq j$). Now for $0 \leq j \leq p_i - 1$ let

$$(y_{i,j}, y_{i,j+1}) \in E_i \text{ with } l_i(y_{i,j}, y_{i,j+1}) = b$$

and for $0 \leq j \leq k_i - 1$ let

$$\begin{aligned} (z_{i,j}, z_{i,j+1}) &\in E_i \text{ with } l_i(z_{i,j}, z_{i,j+1}) = u_{i,j+1} && \text{if } u_{i,j+1} \in X \\ (z_{i,j+1}, z_{i,j}) &\in E_i \text{ with } l_i(z_{i,j+1}, z_{i,j}) = u_{i,j+1}^{-1} && \text{if } u_{i,j+1}^{-1} \in X \end{aligned}$$

Since $l_i(x_i, y_{i,1}) = b \neq a = l_i(x_i, z_{i,1})$, and every other vertex has in- and out-degree at most 1, G_i is indeed an X -labeled graph.

Let Φ_i be the action of F_X on V_i defined by G_i . Then the assumptions of Lemma 4 hold for G_i and w_i with $v_0 = x_i$, so we get

$$x_i \Phi_i(w_i) = z_{i,k_i} \neq x_i$$

For $n \geq 1$ let

$$U_n = \bigoplus_{i=1}^n V_i, \quad o_n = \bigoplus_{i=1}^n x_i \text{ and } \Psi_n = \bigoplus_{i=1}^n \Phi_i$$

Let O_n be the orbit of o_n under Ψ_n and let H_n be the stabilizer of o_n in F_X . Then (H_n) is a chain in F_X and the right coset action of F_X on F_X/H_n is equal to the restriction of Ψ_n to O_n .

Let

$$U_n = \bigoplus_{i=1}^n Y_i \text{ and } P_n = \bigoplus_{i=1}^n (Y_i \setminus \{x_i\})$$

We claim that $U_n \subseteq O_n$. Indeed, $o_n \in U_n$ and $b \in F_X$ acts on Y_i as a cycle of length p_i . Since $\gcd(p_i, p_j) = 1$ ($i \neq j$), the orbit of o_n under the cyclic group $\langle b \rangle$ equals U_n . By the definition of Φ_i , a fixes P_n pointwise, implying

$$\text{fixr}(a, \Gamma/H_n) \geq \frac{|P_n|}{|O_n|} \geq \frac{\prod_{i=1}^n (p_i - 1)}{\prod_{i=1}^n (p_i + k_i)} = \prod_{i=1}^n \left(1 - \frac{k_i + 1}{p_i + k_i}\right) > \alpha$$

Using Lemma 3 this implies that the boundary representation of Γ with respect to (H_n) is not essentially free.

Of course, $\cap_n H_n$ is not necessarily trivial. In fact, if $d > 2$ then c_1 fixes o_n ($n \geq 1$) so $c_1 \in \cap_n H_n$. We claim however, that the boundary representation of Γ with respect to (H_n) is faithful. Assume it is not. Let $w \in \Gamma$ ($w \neq 1$) be an element of the kernel. Then either w or w^{-1} is conjugate to w_m for some m . But then $x_m \Phi_m(w_m) \neq x_m$, implying $o_m \Psi_m(w_m) \neq o_m$. So w_m is not in the kernel of Ψ_m , a contradiction. The claim holds.

Now we invoke a result in [1] saying that every faithful boundary representation of a countable free group is topologically free. We get that there exists $x \in \partial T$ with trivial stabilizer in Γ . Let t_i be the vertex lying on the ray x of level i and let $\Gamma_i = \text{Stab}_\Gamma(t_i)$. Then $\cap_i \Gamma_i = 1$ and since Γ acts transitively on each level of T , Γ_i is conjugate to H_i in Γ ($i \geq 0$). This implies that the boundary action of Γ with respect to (Γ_i) is isomorphic to the boundary action with respect to (H_i) . In particular, the boundary action of Γ with respect to (Γ_i) is not essentially free. The theorem is proved. \square

On a problem of Grigorchuk, Nekrashevich and Sushchanskii. Now we show how this leads to the solution of [4, Problem 7.3.3]. Fix $d \geq 2$ and a minimal generating set X in F_d . Let us take the chain constructed in Theorem 1. Then the action of F_d on the coset tree is spherically transitive and topologically free, that is, a Baire generic point of the boundary has trivial stabilizer in F_d . This implies that the Schreier graph of an orbit of a Baire typical point is isomorphic to the Cayley graph $\text{Cay}(F_d, X)$, which is an infinite $2d$ -regular tree. On the other hand, the boundary representation is not essentially free, which implies that the Schreier graph of an orbit of a measure typical point is isomorphic to the Schreier graph $\text{Sch}(F_d/H, X)$ where H is a nontrivial subgroup of F_d . This graph is hence also $2d$ -regular but is never a tree. Thus there exists a ball in $\text{Sch}(F_d/H, X)$ which can not be embedded into $\text{Cay}(F_d, X)$.

Remark. Our method provides a chain in F_n that has rapidly growing index or, equivalently, the number of children of a vertex on the coset tree grows very fast. However, there is no reason to assume that this is necessary. In fact, an infinite rooted binary tree should be sufficient. It may be possible to use a random method to show that an odometer (a.k.a. adding machine) and a

random element fixing a fixed nowhere dense set of positive measure on the boundary of a rooted binary tree generates a free group a.s.

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